A Deeper Look at a Calculus I Activity

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Abstract

As a result of NSF-funded course redesign efforts to implement, promote and research active learning in introductory calculus, this paper discusses a derivative sketching activity for Calculus I. Pilot research indicates that overly-qualitative approaches to the activity often lead to certain incorrect student graphs. After revealing a deeper look at the mathematics behind the activity, the paper explores an approach to moderating the lesson in a way that leads students to a deeper understanding by activating familiar pre-requisite knowledge, without requiring mathematics beyond their zones of proximal development.

Introduction

This brief paper focuses on one course redesign approach for first-year calculus resulting from collaborations with the Mathematics FLOK (Faculty Learning for Outcomes and Knowledge) group at Fresno State. Key elements of this redesign philosophy are based on two principles inspired from mathematics education literature as well as writings in cognitive psychology and research on analogical transfer in learning (Harel, 2007):

1. The Necessity Principle: “For students to learn what we intend to teach them, they must have a need for it, where ‘need’ refers to intellectual need, not social or economic need.” (pp. 275-276)
2. The Repeated Reasoning Principle: “Students must practice reasoning in order to internalize, organize, and retain ways of understanding and ways of thinking.” (pp. 275-276)

The above principles influence the *what* and *how* topics are covered in this reform classroom. In terms of implementation, these two principles have taken form in the following recommendations for course redesign.

- Review of prerequisite material should be avoided.
- Important ideas and problem solving should commence as soon as possible so that their practice can induce recognition of patterns to problem solving.

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- Active learning is essential for students to authentically internalize, apprehend and communicate mathematics.

To understand the redesign rationale from another viewpoint, suppose that an enduring an idea such as the derivative concept is viewed as a bicycle. Clearly there are many components, yet looking at them in isolation and adding more and more components to the picture does not provide a bicycle until the parts list has been completed (see Fig. 1).

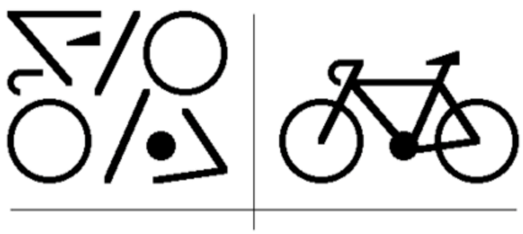


Figure 1. Learning to ride a bicycle would be near impossible from just handling the parts (I Think in Pictures, 2010).

A *Wholecept* is a cognitive structure, arrangement, or pattern of mathematical phenomena so integrated as to constitute a functional unit with properties not derivable by summation of its parts.

The Wholecept definition was originally inspired from Tall's "precept" notion which blurs the distinction between processes and concepts; but as reflection on teaching was refined, Fritz Perls' gestalt therapy writings informed the need for a dynamic element similar to the gestalt foreground/background process of conflict resolution (Grey & Tall, 1994; Perls, 1973). According to Perls, if cognitive difficulty is in the foreground, then one cannot proceed until the difficulty is resolved and made to retreat to the background so that progression can be made to deeper conflict resolution (see Fig. 2). In this respect, real conflict in student learning is not due to lack of understanding of prerequisite material, but rather to the need for a coherent picture of the "relevance" of any particular mathematical topic they are being required to learn; hence, the above recommendations.

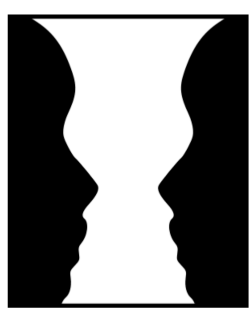


Figure 2. Rubin's (2001) famous illustration of figure-ground perception.

To restate this in terms of Harel (2007), violation of the Necessity Principle constitutes a fundamental roadblock to learning. The wholecept represents mathematical knowledge that is more "found" than constructed through a dynamic process of gradual conflict resolution and discovery. In this sense, the philosophical underpinnings of this emergent theory of mathematical learning have Platonist underpinnings, rather than being purely a constructivist view of learning. Mazur (2008) elegantly captures this viewpoint in the following quote:

> When I'm working I sometimes have the sense—possibly the illusion—of gazing on the bare platonic beauty of structure or of mathematical objects, and at other times I'm a happy Kantian, marveling at the generative power of the intuitions for setting what an Aristotelian might call the formal conditions of an object. And sometimes I seem to straddle these camps (and this represents no contradiction to me). I feel that the intensity of this experience, the vertiginous imaginings, the leaps of intuition, the breathlessness that results from "seeing" but where the sights are of entities abiding in some realm of ideas, and the passion of it all, is what makes mathematics so supremely important for me. (p. 20)

Figure 3 describes the conventional approach to apprehending a wholecept, such as the *derivative wholecept*, by building up from the basics, linearly, until the derivative can eventually be defined and examples can finally begin which employ and connect the previously learned material to the main topic. A central weakness of this approach is that students often have very little time practicing problems, reasoning and communicating ideas related to the "big picture," which can contribute to poor exam performance and retention of the material. This is represented by the faintness of the final large circle in Figure 3.

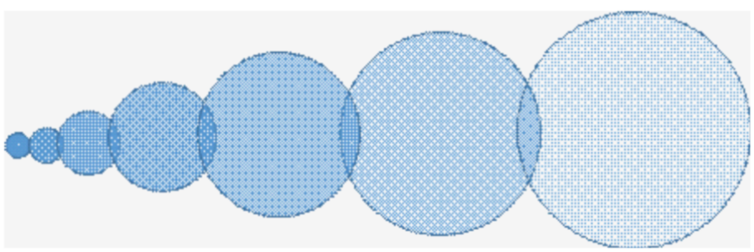


Figure 3. Unilinear concept formation—learning to ride a bike by building it one piece at a time and then trying to ride only when completed.

In stark contrast, Figure 4 depicts a very faint initial picture of the entire wholecept which, by repetition, becomes more and more clear to a point of eventual mastery. Note that the Final circle in Figure 4 is as dark as the smallest low-level circle in Figure 3, implying that the derivative wholecept has now become a functional unit applicable to a much larger picture.

To illustrate how these ideas could be applied to Calculus I, one could begin with the derivative wholecept in its entirety on the first day of class, and then continually pull in the "necessary" concepts which are needed to make it work, so to speak, so that rich problems arising from the derivative wholecept can begin and repeated as soon and as long as possible.

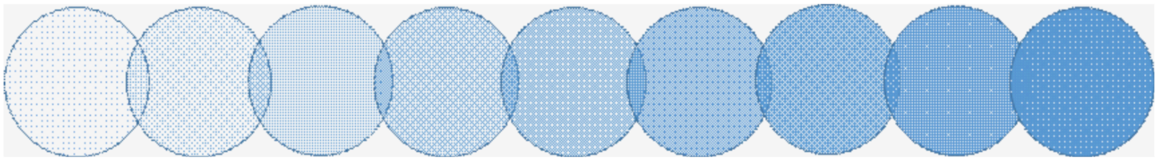


Figure 4. Wholecept resolution—taking a longer time to repetitively learn to ride a functioning bike.

Through this repetitive mantra of rich-structure problem solving, concepts such as the one-sided and two-sided limit, continuity, graphs, slopes, functions and tangent lines start to have renewed meaning. Further, this allows for the student to resolve issues of content relevancy, which may now retreat to the background, so that connections can be recognized and larger-scale problem solving patterns practiced and learned. Next, an activity in derivative sketching is discussed which is one of many weekly activities used in the infusion of active earning in calculus at Fresno State, in collaboration with the Boulder-Omaha Active Learning Alliance (2015).

Activity Description

The initial problem: *Coffee is being poured at a constant rate* v *into coffee cups of various shapes. Sketch rough graphs of the rate of change of the depth* $h'(t)$ *and of the depth* $h(t)$ *as a functions of time t (see Fig.5).*

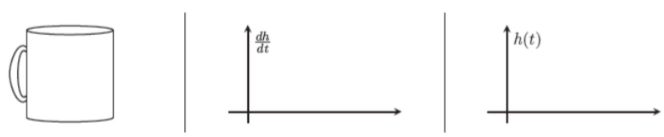


Figure 5. The cylindrical cup.

The two cup shapes discussed in this paper are the cylindrical and frustum shaped cups. In informal terms, most all students over three semesters of implementation produce qualitatively correct graphs for the straight-sided cup (see Fig. 6).

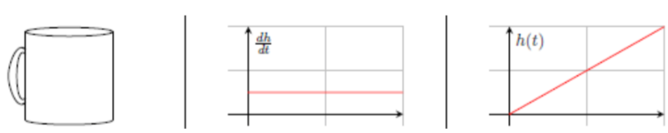


Figure 6. Cylindrical cup student solutions.

Slant-sided cup. In contrast, for the inverted frustum cup most all students produce incorrect graphs for $(t, \frac{dh}{dt})$ (see Fig. 7). In the following section, the mathematics behind these related rate graphs is discussed; however, it should be emphasized that the students participating in this activity are not expected to understand it at the depth to be discussed. An important aim of the mathematical treatment given in this paper, though, is to caution against overly qualitative approaches when a deeper understanding of the mathematics behind an activity can greatly inform pedagogy.

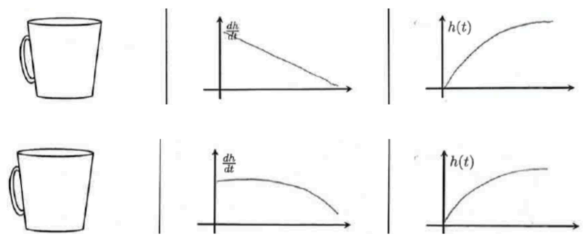


Figure 7. Typical student solutions of slant-sided cup.

A Deeper Look

Looking more closely at the cylindrical cup with base radius r_0, we can safely conclude that since the volume $V(t)$ of coffee in the cup increases at a constant rate, then so does its depth. Hence, $h'(t) \equiv\ h$ and $h(t)\ =\ ht$ (the cup being empty initially, i.e., $h(0)\ =\ 0$).

$$V(t) = \pi r_0^2 h(t)$$

Differentiating both sides relative to t

$$V'(t) = \pi r_0^2 h'(t)$$

and considering that $V'(t) = v$, we have: $h'(t) = \frac{v}{\pi r^2} \longrightarrow h(t) = \frac{v}{\pi r_0^2} t$ (given $h(0) = 0$).

As seen in Figure 6, the typically correct student graphs align well with the mathematics, since $(t, h'(t))$ produces a constant function horizontal graph, and $(t, h(t))$ consists of a linear graph through the origin with positive slope. Observe that $h'(t)$ is not the same as $V'(t)$, although this fact may elude students' attention when only a qualitative approach is applied.

For the slant-sided (inverted frustum) cup, let $r(t)$ be the radius of the surface of coffee. Then

$$r(t) = r0 + mh(t)$$

with some $m > 0$.

In this case, it appears "natural" to think of $h'(t)$ as a linear function based on the linear dependence of the radius $r(t)$ on the depth $h(t)$ which leads to the conclusion that $h'(t)$ is a *linear function* and $h(t)$ is *quadratic*. But as we shall see, this described qualitative approach fails the test by mathematics since by the conical frustum volume formula, the volume of coffee in the cup at time t is given by:

$$V(t) = \frac{1}{3}\pi[r_0^2 + r_0 r(t) + r^2(t)]h(t).$$

Instead of differentiating both sides of the above equation relative to t, which would make things more convoluted, we consider that $V'(t) \equiv v$ immediately implies $V'(t) = vt$ (with $V(0) = 0$); hence, $h(t)$ is to be found from the cubic equation:

$$m^2 h^3(t) + 3mr_0 h^2(t) + 3r_0^2 h(t) - 3vt/\pi = 0.$$

As recalled in texts such as Boyer and Merzbach (1991), the general formula for the roots of such an equation in this case yields $h(t)$ explicitly as

$$h(t) = -\frac{1}{3m^2}\left[3mr_0 + \sqrt[3]{-27m^3 r_0^3 - 81m^4 vt/\pi}\right].$$

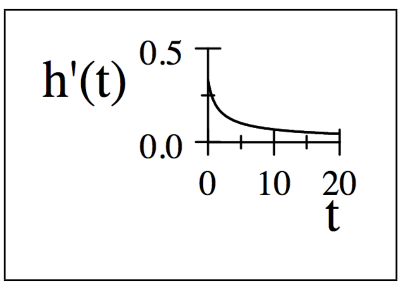
Hence

$$h(t) = a(t+b)^{1/3} + c$$

with some $a; b > 0$ and $c < 0$ such that $h(0) = ab^{1/3} + c = 0$ and

$$h'(t) = \frac{a}{3}(t+b)^{-2/3}. \text{ Eq.[1]}$$

Letting $a = b = 1$ and $c = 1$ satisfies the initial conditions and produces qualitatively accurate graphs for $h(t)$ and $h'(t)$ (see Fig. 8)



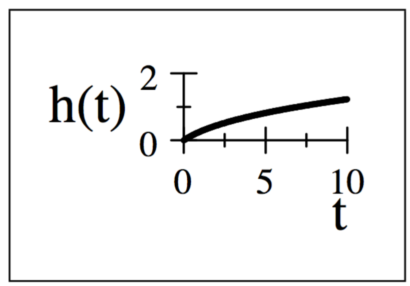


Figure 8. Frustum cup graphs.

The Exponential-Sided Cup. As a Calculus II extension of the previous analyses, the disk method performed on an exponential-sided cup highlights the mathematical depth lying behind this activity when analyzing vessels which are widening (or narrowing) (see Fig. 9).

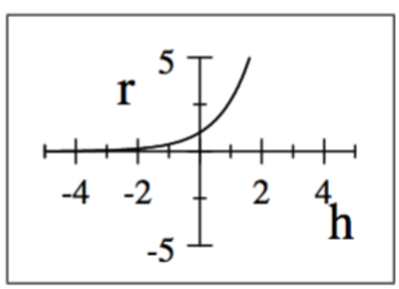


Figure 9. Flat-bottom exponential-sided cup generated by revolving $y = e^x$ around the x axis $x \to 0$ to $h > 0$.

Using the disk-method from 0 to h and employing the previous technique letting $V'(t) \equiv\ v$ and $V(t) = vt$,

$$V(t) = \pi \int_0^h (e^x)\, dh = \frac{\pi e^{2h}}{2} - \frac{\pi}{2} = vt.$$

whereby solving for h gives

$$h(t) = \frac{1}{2} ln\left(\frac{2vt}{\pi} + 1\right).$$

For a simpler picture, let $v = \frac{\pi}{2}$ which becomes

$$h(t) = \frac{1}{2} ln(t + 1)$$

and then differentiating both sides relative to t we have

$$h'(t) = \frac{1}{2(t + 1)}$$

resulting in graphs qualitatively similar to the slant-sided cup graphs (see Figs. 8 & 10).

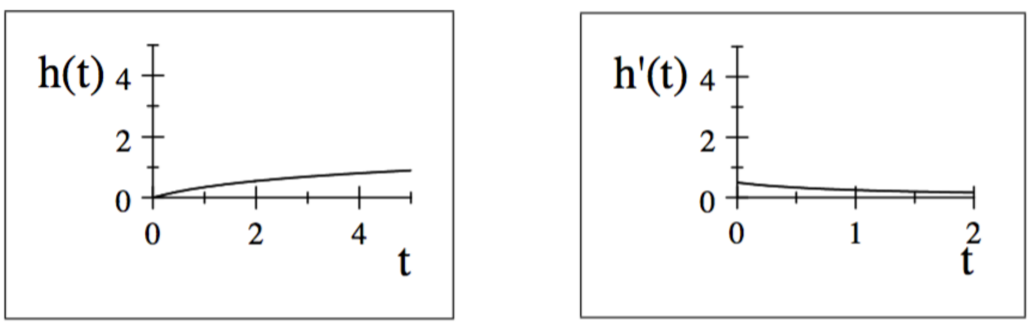


Figure 10. Exponential-sided cup graphs.

Facilitating Transfer: Known to Unknown

A first place to start when debriefing groups of students on this activity can begin with collaborative discussions about their interpretations of their graphs. For example, looking back at the slant-sided student solution graphs (see Fig. 7), after some good questioning students can arrive at the conclusion that the incorrect graphs $(t, h'(t))$ don't make sense since they imply that the rate of change of the height eventually becomes 0. A fact contradicting the constant filling of the cup, and moreover if continued, $\frac{dh}{dt}$ becomes negative implying the height function is decreasing.

On a positive note, students can also reflect on the fact that their $(t, h(t))$ graphs usually do make sense, since they start at 0 and increase, yet the rate of increase slows down as seen by the tangent lines to the graph becoming more horizontal and approaching zero, consistent with the assumption of constant filling of an increasingly widening cup of coffee. So the question remains, how can students arrive at correct $(t, h'(t))$ graphs given the mathematics they know? *Analogical problem construction (APC)* refers to "letting students construct their own analogous problems,… [which] allows the problem solver to use his or her own knowledge and experiences to create the analogical problem elements" (Bernardo, 2001, p. 138). In a mathematics study on APC, Bernardo (2001) found that,

> One can use a rather structured task, and still allow students to explore and engage the information in math problems enough to lead them to deeper levels of understanding of the problems which increase analogical transfer performance. (pp. 147-148)

This paper concludes with some structured examples for how APC can be induced in the context of this activity.

Conclusion

Promoting analogical problem construction in the context of this calculus activity can begin by asking students to collaboratively produce familiar functions that resemble their $(t, h(t))$ graphs. After discussion and concensus, they can be asked to find the derivative graphs of these familiar functions and compare them to those made in the cup activity. As an example, the following two functions are familiar to most students and have graphs that match the initial conditions and have the same qualitative shapes as their correctly produced $(t, h(t))$ graphs:

- $h(t) = \sqrt{t}$
- $h(t) = ln(t + 1)$

At this point in the course material, calculus students can easily take these derivatives and sketch their graphs (see Figs. 11 & 12), and then compare them to the ones they produced. Important topics such as concavity can be discussed as well as subtleties, such as the difference between Figures 8 and 11, where in Figure 11 the $(t, h'(t))$ graph appears to be infinite at $t = 0$; illustrating the degenerate case when the frustum is a cone (see Eq.[1], and consider when $b = 0$).

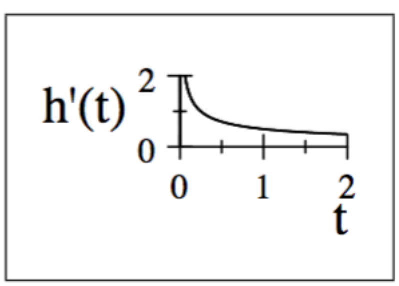


Figure 11. $(t, \frac{1}{2\sqrt{t}})$ graph.

As this activity is done at the end of the semester, when anti-differentiation has been covered, the previous line of questioning involving graphing the derivative of familiar functions of $(t, h(t))$ can be reversed to the case of finding familiar functions to their $(t, h'(t))$ graphs, and exploring problematical issues associated with graphs of their anti-derivatives.

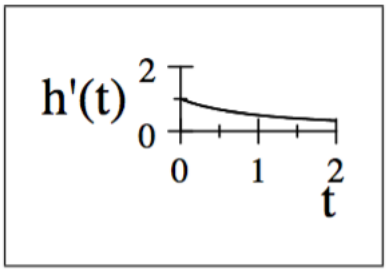


Figure 12. $(t, \frac{1}{t+1})$ graph.

For example, the following two functions have the same qualitative shapes as their typically *incorrect* $(t, h'(t))$ graphs (see Fig. 7):

- $h'(t) = -2t + 3$
- $h'(t) = -t^2 + 2$

Recalling the initial condition that $h(0) = 0$ then for both antiderivatives $C = 0$; hence,

$$\int -2t + 3dt = -t^2 + 3t + C = -t^2 + 3t$$

$$\int -t^2 + 2dt = \frac{t^3}{3} + 2t + C = \frac{t^3}{3} + 2t$$

The anti-derivitive computations produce the above non-sensical graphs, which may promote rich discussions as they are problematical for a variety of reasons, one being they imply the height increases then decreases, again contradicting the assumption of constant filling of the coffee cups, (see Figs. 13, 14).

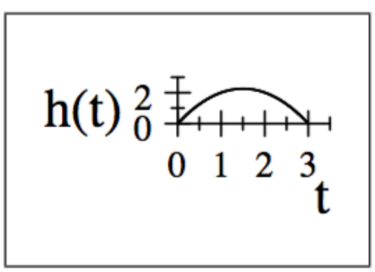


Figure 13. Anti-derivative graph for $h'(t) = -2t + 3$.

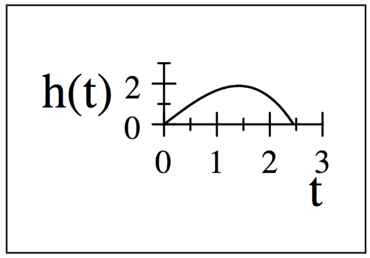


Figure 14. Anti-derivative graph for $h(t) = -t^2 + 2$.

In summary, although active learning can and should involve fun, interactive and concrete ways to explore mathematical concepts, a deeper understanding and exploration of the underlying mathematics by the instructor should not be avoided, as it can hold the keys to unlocking latent student knowledge already lying dormant within.

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